ON THE CONVECTIVE INSTABILITY OF A THERMAL BOUNDARY LAYER

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When the surface temperature of a liquid is a harmonic function of time with a frequency ω, a temperature wave propagates into the liquid. The amplitude of this wave decreases exponentially with distance from the surface. The temperature oscillation is essentially concentrated in a layer of the order of $(2\chi / \omega)^{1/2}$, where χ is the thermal conductivity of the liquid (thermal boundary layer). Depending on the phase, at certain positions below the surface the temperature gradient is directed downwards and if its magnitude is sufficiently large (the magnitude is a function of the amplitude and frequency of the surface oscillations) the liquid can become unstable with respect to the onset of convection. In that case the convective motion may spread beyond the initial unstable layer. For low frequencies the stability condition can be derived from the usual static Rayleigh criterion, on the basis of the Rayleigh number and the average temperature gradient of the unstable layer. This quasi-static approach, used by Sal'nikov [1], is appropriate to those cases in which the period of the temperature oscillations is much larger than the characteristic time of the perturbations. But when these times are of the same order, the problem must be analyzed in dynamic terms. The stability problem must then be formulated as a problem of parametric-resonance excitation of velocity oscillations due to the action of a variable parameter—the temperature gradient.

In an earlier work [2] we considered the problem of the stability of a horizontal layer of liquid with a periodically varying temperature gradient. It was assumed that the thickness of the layer was much smaller than the penetration depth of the thermal wave, so that the temperature gradient could be assumed to be independent of position. In the present work we consider the opposite case, in which the liquid layer is assumed to be much larger than the penetration depth, i. e., a thermal boundary layer can be defined. The temperature gradient at equilibrium, which is a parameter in the equations determining the onset of perturbations, is here a periodic function of time and a relatively complicated function of the depth coordinate z. The periodic oscillations are solved by the Fourier method; the equations for the amplitudes are solved by the approximate method of Karman-Pohlhausen.

1. The equations of natural convection are, in the usual notation, [3]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla p + \nu\Delta\mathbf{v} - \mathbf{g}\beta T,$$
$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \chi\Delta T, \qquad \operatorname{div}\mathbf{v} = 0. \tag{1.1}$$

We choose our coordinates so that the (x, y) plane coincides with the liquid surface. The z axis is directed downwards. The temperature of the surface z = 0 is uniform over the surface and varies with time as a harmonic function

$$T_0 = \Theta\cos\omega t. \tag{1.2}$$

At equilibrium ($\mathbf{v} = 0$) the temperature inside the liquid is determined from the equation

$$\frac{\partial T_0}{\partial t} = \chi\frac{\partial^2 T_0}{\partial z^2}. \tag{1.3}$$

This has the solution

$$T_0 = \Theta e^{-\varkappa z}\cos(\omega t - \varkappa z), \qquad \varkappa = (\omega/2\chi)^{1/2}, \tag{1.4}$$

which satisfies (1.2) and vanishes at $z \to \infty$ (we choose the temperature of the liquid at a large depth below the surface as the temperature datum). The quantity $\delta = 1/\varkappa$ may be regarded as a measure of the depth of penetration of the thermal wave. The equilibrium pressure p_0 is determined by the equation

$$\nabla p_0 = -\rho_0\beta T_0\mathbf{g}. \tag{1.5}$$

To investigate the stability of the unsteady equilibrium (T_0, p_0) we consider small convective perturbations ($T', p', \mathbf{v}$). Substituting the perturbed temperature and pressure $T_0 + T'$, $p_0 + p'$ and the velocity $\mathbf{v}$ into the system (1.1) and retaining the linear terms, we obtain a system of equations for the perturbations

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0}\nabla p' + \nu\Delta\mathbf{v} - \mathbf{g}\beta T',$$
$$\frac{\partial T'}{\partial t} + v_z\frac{\partial T_0}{\partial z} = \chi\Delta T', \qquad \operatorname{div}\mathbf{v} = 0. \tag{1.6}$$

We can eliminate the variables p', v_x, v_y by applying to the first equation in (1.6) the operator curl curl and projecting the result on the z axis. Further, we assume that the perturbations are periodic in the (x, y) plane, i.e. all variables are proportional to $\exp i(k_1x + k_2y)$. The vertical velocity component $v_z \equiv v$ and the temperature perturbation T' (in the following we shall drop the prime) are determined by the equations

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta v = g\beta k^2 T, \qquad \left(\frac{\partial}{\partial t} - \chi\Delta\right)T = -\frac{\partial T_0}{\partial z}v,$$
$$\left(k^2 = k_1^2 + k_2^2, \qquad \Delta = \frac{\partial^2}{\partial z^2} - k^2\right). \tag{1.7}$$

We rewrite (1.7) in dimensionless form, choosing the units of length, time, speed, and temperature to be $1/\varkappa$, $1/\varkappa^2\sqrt{\nu\chi}$, Θ, $g\beta k^2\Theta/\nu\varkappa^4$. This yields

$$\left(\frac{1}{\sqrt{P}}\frac{\partial}{\partial t} - \Delta\right)\Delta v = T, \quad \left(\sqrt{P}\frac{\partial}{\partial t} - \Delta\right)T = -Rk^2 f(z, t)\,v$$
$$(k = (k_1^2 + k_2^2)^{1/2}/\varkappa, \quad P = \nu/\chi, \quad R = g\beta\Theta/\nu\chi\varkappa^3). \tag{1.8}$$

Here k is a dimensionless wave-number, P is the Prandtl number, R is the Rayleigh number based on the temperature amplitude at the surface Θ and the penetration depth $1/\varkappa$, and $f(z, t)$ is the dimensionless temperature gradient at equilibrium,

$$f(z, t) = e^{-z}\left[\sin\left(\frac{2t}{\sqrt{P}} - z\right) - \cos\left(\frac{2t}{\sqrt{P}} - z\right)\right], \tag{1.9}$$

(here t and z are dimensionless).

The perturbations v and T must decay exponentially for $z \to \infty$. As regards the boundary conditions at the upper surface, we shall consider two cases:

a) **Plane free surface.** The normal component of velocity, the tangential stresses, and the temperature perturbation vanish, i.e.

$$v = \frac{\partial^2 v}{\partial z^2} = T = 0 \quad \text{at } z = 0. \tag{1.10}$$

b) Solid boundary. All velocity components and the temperature perturbation vanish, i.e.,

$$v = \frac{\partial v}{\partial z} = T = 0 \quad \text{at } z = 0. \tag{1.11}$$

2. We shall consider the free-surface case first. Eliminating the temperature perturbation T from the system (1.8), we obtain an equation for v

$$\left(\frac{\partial}{\partial t} - \sqrt{P}\Delta\right)\left(\frac{\partial}{\partial t} - \frac{1}{\sqrt{P}}\Delta\right)\Delta v = -k^2 R f(z, t)\, v. \tag{2.1}$$

Taking into account (1.10) and (1.8), we replace the boundary condition T = 0 at z = 0 by the condition

$$\frac{\partial^4 v}{\partial z^4} = 0 \quad \text{at } z = 0. \tag{2.2}$$

To find the stability criterion, we must find a relation between the parameters R, P, k for which there exists a solution of (2.1) which is periodic in time. As is well known, the main region of parametric resonance corresponds to motion with a frequency equal to one half of the excitation frequency. Therefore we may seek a "half-integer" periodic solution of (2.1) in the form

$$v = v_1(z)\cos\frac{t}{\sqrt{P}} + v_2(z)\sin\frac{t}{\sqrt{P}} + \cdots \tag{2.3}$$

using the Fourier method. Here $v_1(z)$ and $v_2(x)$ are the amplitudes of the basic harmonic with the frequency $1/\sqrt{P}$. The upper harmonics, which we have not written out here, have the frequencies $3/\sqrt{P}$, $5/\sqrt{P}$;.... Substituting (2.3) into (2.1) and retaining only the basic harmonic, we obtain a system of ordinary homogeneous differential equations for the Fourier-amplitudes $v_1(z)$ and $v_2(z)$:

$$P\Delta^3 v_1 - (1 + P)\,\Delta^2 v_2 - \Delta v_1 = k^2 PR\,(\varphi_+ v_1 - \varphi_- v_2),$$

$$P\Delta^3 v_2 + (1 + P)\,\Delta^2 v_1 - \Delta v_2 = k^2 PR\,(\varphi_- v_1 + \varphi_+ v_2),$$

$$(\varphi_\pm = {}^1/_2\, e^{-z}(\cos z \pm \sin z), \quad \Delta = d^2/dz^2 - k^2). \tag{2.4}$$

The amplitudes v_1 and v_2 satisfy the homogeneous boundary conditions

$$v_{1,2} = v_{1,2}'' = v_{1,2}^{\text{IV}} = 0 \quad \text{at } z = 0,$$

$$v_{1,2} \to 0 \quad \text{at } z \to \infty. \tag{2.5}$$

To find an approximate solution of the boundary-value problem (2.4), (2.5) we can use the Karman-Pohlhausen method, used in boundary-layer theory [4]. According to this method, the solution is approximated by an assumed expression which takes account of the boundary conditions, and the parameters in that expression are then determined from integral relations. In our case the boundary conditions for v_1 and v_2 are identical, so that in the first approximation, involving the minimum number of parameters, we may assume

$$v_1(z) = c_1 F(z), \qquad v_2(z) = c_2 F(z) \tag{2.6}$$

with a common function

$$F(z) = (3z + 3az^2 + a^2 z^3)\, e^{-az} \tag{2.7}$$

which satisfies (2.5). Here a is a parameter which characterizes the depth of penetration of the perturbations.

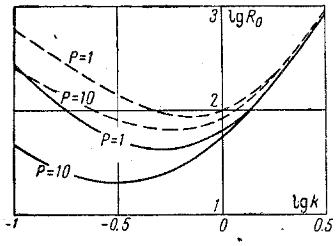

Fig. 1

Substituting (2.6) into (2.4) and integrating with respect to z from 0 to ∞, we obtain a system of homogeneous algebraic equations for c_1 and c_2. The compatibility condition of this system yields a relation between the parameters

$$R^2 = \frac{(I_1 - PI_3)^2 + (1 + P)^2 I_2^2}{a^4 k^4 P^2 K^2},$$

$$\text{where } \left(K^2 = \frac{a^2(5a + 4)^2 + 4(2a + 1)^2}{2\,[(a + 1)^2 + 1]^4}\right)$$

$$I_1 = a^2 + 5k^2, \; I_2 = a^4 + 2k^2 a^2 + 5k^4,$$

$$I_3 = 5a^6 + 3k^2 a^4 + 3k^4 a^2 + 5k^6. \tag{2.8}$$

An analogous relation can be derived for the case of a solid boundary. In that case it is more convenient to use (1.8). The Fourier expansion of the "half-integer" periodic solution of (1.8) begins with the harmonics

$$v = v_1(z)\cos\frac{t}{\sqrt{P}} + v_2(z)\sin\frac{t}{\sqrt{P}} + \cdots,$$

$$T = T_1(z)\cos\frac{t}{\sqrt{P}} + T_2(z)\sin\frac{t}{\sqrt{P}} + \cdots. \tag{2.9}$$

Substituting (2.9) into (1.8) and neglecting the higher harmonics, we obtain a system of homogeneous ordinary differential equations for the amplitudes v_1, v_2, T_1, and T_2. For the sake of brevity we do not write it out. The approximations for the amplitudes are now

$$v_1 = c_1 F(z), \quad v_2 = c_2 F(z),$$

$$T_1 = d_1 \Phi(z), \quad T_2 = d_2 \Phi(z) \tag{2.10}$$

where

$$F(z) = z^2 e^{-az}, \qquad \Phi(z) = (z + az^2)\, e^{-az}. \tag{2.11}$$

This choice of F(z) and Φ(z) satisfies the boundary conditions

$$v_{1,2} = v_{1,2}' = 0, \quad T_{1,2} = 0 \quad \text{at } z = 0;$$

$$v_{1,2},\, T_{1,2} \to 0 \quad \text{at } z \to \infty \tag{2.12}$$

as well as the additional condition

$$T''_{1,2} = 0 \quad \text{at } z = 0 \tag{2.13}$$

which follows from (1.8).

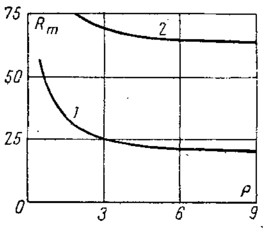

Fig. 2

Integrating the amplitude equations over z from 0 to ∞ and taking account of (2.10), the compatibility condition of the system for c_1, c_2, d_1, and d_2 yields a relation analogous to (2.8):

$$R^2 = \frac{2\,[(a+1)^2+1]^3}{9k^4a^6P^2}\,[(I_1I_2P - 3k^2)^2 + (3I_1P + I_2k^2)^2]$$

$$I_1 = 3a^4 + k^4, \qquad I_2 = a^2 + 3k^2. \tag{2.14}$$

3. The relations (2.8) and (2.14) contain the still undetermined parameter a. Unlike the wave-number k, this parameter cannot be specified independently but is determined by the other parameters of the problem. To determine this parameter we can use any other integral relation, e.g., the momentum equation. In principle, such an approach would yield the parameter a and then one could use (2.8) and (2.14) to determine the relation between R and k at the stability limit ("neutral" curve) for the two cases considered. This, however, leads to very cumbersome relations, not easily amenable to analysis. Therefore we had to restrict our analysis to the determination of the lower bound of the region of instability.

One can see from (2.8) and (2.14) that for fixed P and k the function $R(a)$ has a minimum at some point a_0. This value a_0 cannot, however, be regarded as a true characteristic measure of the penetration depth. But the corresponding minimum value R_0 is a lower bound of the region of instability for fixed P and k, i.e. essentially a lower bound for the amplitude Θ necessary for the onset of convection for the given values of frequency and wavelength.

Figure 1 shows the dependence of R_0 on k for various values of the Prandtl number. The solid curves correspond to the case of a free surface; the dashed line corresponds to a solid boundary. The minimum of the $R_0(k)$ curve determines the critical perturbation wavenumber k_m and the critical values R_m for a solid boundary are higher than the corresponding values for a free surface. For $k \gg 1$ the stability curves for all Prandtl numbers coincide. Moreover, for large k the values R_0 are independent of the boundary conditions.

When the Prandtl number increases, the critical Rayleigh number decreases and the minimum moves in the direction of long-wave perturbations (Figs. 2 and 3; 1 and 2 correspond to a free surface and a solid boundary, respectively).

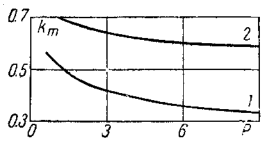

Fig. 3

For water at 20° C (P = 7), for instance, the critical R yleigh number for a free surface is R_m = 20. Hence the condition for the onset of convection

$$\Theta > 20\,\frac{\nu\chi}{g\beta}\left(\frac{\omega}{2\chi}\right)^{3/2}.$$

For a period of 1 min the critical amplitude is Θ > 0.4° C.

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